Fredy & Nadine Herrmann Institute of Earth Sciences

# Mathematical Methods In Scientific Models

Fall Semester - 2020

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# Chapter 1

# Small is NOT necessarily negligible

The standard practice in natural sciences to understand any physical, chemical or biological system is to somehow express it in the form of an equation. The equation to find the displacement $s$ of a particle traveling along a straight line at a speed $u$ and accelerating with time at a rate $a$ is given by:

$$s = ut + \frac{1}{2}at^2 \tag{1.1}$$

where $t$ is the time lapsed since the particle started moving. On the other hand, the decay of a radio active element like uranium can be quantified as:

$$N(t) = N_0 e^{-kt} \tag{1.2}$$

where $N_0$ is the initial number of atoms of uranium, $k$ is the decay constant and $N(t)$ is the number of uranium atoms present after time $t$.

Such equations are a scientist's attempt to express any natural phenomena in terms of simple mathematical expressions often aimed at predicting the future state of any system. The general form of an equation is given by $f(x) = 0$. The left hand side (LHS) of the equation can consist of any number of terms. In context of determining the future state of a natural system, the terms on the LHS usually represent different mechanisms governing the system.

## 1.1 Finding approximate solutions to equations (or problems) that have no simple solutions

Ever so often, the scientific problems that we express as equations are not solvable. In that case one has to resort to some 'trickery' (a protocol) which makes the problem solvable. One such protocol which will be used extensively throughout this course can be summarised as follows:

1. Identify a (seemingly) small term in the problem/equation.
2. Solve the problem by setting the "small" term equal to 0.
3. Verify that the small term is indeed small by substituting the approximate solution we've found in the original problem and checking that the neglected term is indeed small.

The third step in the protocol ensures an apparent consistency. However, the reader should note that an apparently consistent solution may not be the real solution to the problem. We attempt to understand this protocol through a set of examples. We start the discussion by examining problems with known solutions in order to review the usefulness of the protocol.

### 1.1.1 Example 1: the protocol works

Equations (1.3),(1.4) make up a set of linear algebraic equations with two unknowns. One learns to solve such equations in middle school but for the sake of illustrating our protocol we assume that we do not know how to solve this set of equations exactly.

$$x + 10y = 21 \tag{1.3}$$
$$5x + y = 7 \tag{1.4}$$

The first step of the protocol is to identify a small term, we assume that term $x$ in (1.3) is small based on the values of the coefficients in this equation (1 compared to 10 and 21). As per the second item in our protocol, we set $x = 0$ in which case (1.3) simplifies to $10y \approx 21$ which yields the approximate solution $y = 2.1$. Substituting this value of $y$ in (1.4) then yields the approximate value of $x = 0.98$. The third step is to check the apparent consistency of the solution. Substituting $x = 0.98$ and $y = 2.1$ on the LHS of (1.3) and (1.4) yields 21.98 and 7 respectively. Thus, we see that the condition of apparent consistency is also satisfied since the maximal relative error on the RHS is about 5% (i.e., 0.98/21).

If you solve this set of algebraic equations without following our protocol the exact solution is $x = 1$ and $y = 2$. In this particular case, our protocol has yielded approximate solutions that are close to the exact solutions 2.1 approximates 2 while 0.98 approximates 1 which falls within the anticipated 5% accuracy. However, this is not always the case as we'll see in the next example.

### 1.1.2 Example 2: the protocol does not work

Equations (1.5),(1.6) make up yet another set of linear equations with two variables.

$$0.01x + y = 0.1 \tag{1.5}$$
$$x + 101y = 11 \tag{1.6}$$

Following the protocol, as was done in the previous section, yields that the solution to the problem is $x = 0.9$ and $y = 0.1$. Despite satisfying the condition of apparent consistency, this approximate solution is very far away from the exact solution of $x = -90$ and $y = 1$. We now try to understand why does the protocol fail in this particular case. Set of equations (1.7), (1.8) is a generalization of the above set.

$$\epsilon x + y = 0.1 \tag{1.7}$$
$$x + 101y = 11 \tag{1.8}$$

Here $\epsilon$ is a real number. Solving (1.7) for $x$, yields $x = -\dfrac{0.9}{101\epsilon - 1}$. When $\epsilon \to 1/101$, the value of $x \to -\infty$. The coefficient in (1.5) is very close to $1/101$ and that's why we get spurious solutions while employing the protocol.

### 1.1.3 Example 3: Wilkinson's polynomial

The Wilkinson's polynomial is a polynomial of degree 20, the roots of which are natural numbers $1, 2, 3, \ldots, 20$.

$$(x-1)(x-2)(x-3)\ldots(x-19)(x-20) = 0 \tag{1.9}$$

Let us add a tiny perturbation, $\epsilon \sim 10^{-10}$, to the Wilkinson's polynomial; the perturbed polynomial is given by:

$$(x-1)(x-2)(x-3)\ldots(x-19)(x-20) + \epsilon x^{19} = 0 \tag{1.10}$$

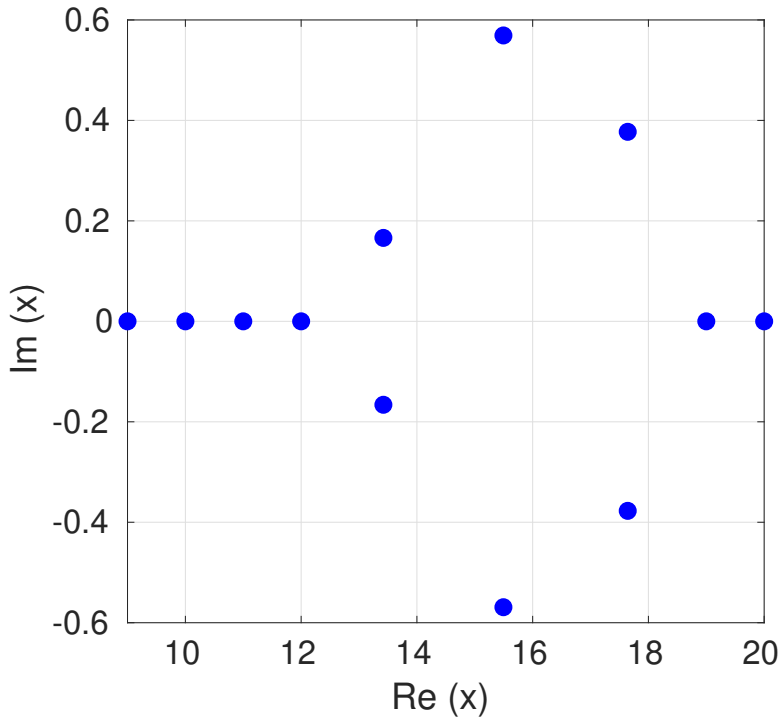


Figure 1.1: Roots of the perturbed Wilkinson's polynomial on the complex plane for $\epsilon = 4 \times 10^{-10}$. Roots smaller than 9 are not shown in the figure. [Adapted from Bender and Orszag (1978)]

As is illustrated in Figure 1.1, even such a 'small' perturbation ($\epsilon = 4 \times 10^{-10}$) renders a few roots of Wilkinson's polynomial (6 in this case), complex. Clearly, we need to develop a precise definition of what can be regarded as small.

## 1.2 Generalization: ill-posed equations/problems

In this section we develop a general framework to answer the question when a quantity appearing in a problem can be considered 'small'. To do the same, we assume that any problem we solve (differential equation, algebraic equation etc) is of the form $f(x) = 0$. The perturbed form of the same can be written as:

$$f[x(\epsilon), \epsilon] = 0. \tag{1.11}$$

We further assume

(a) for $\epsilon = 0$: $f[x(0), 0] = 0$

(b) for $0 < \epsilon \ll 1$: $f[x(0), \epsilon] - f[x(0), 0] \approx \epsilon \frac{\partial f}{\partial \epsilon}\Big|_{x(0)} := r$ is known (this is just the term added to the equation itself).

If $\epsilon$ is indeed small, we can approximate $f[x(\epsilon), \epsilon]$ as:

$$f[x(\epsilon), \epsilon] \approx f[x(0), 0] + \epsilon \frac{\partial f}{\partial x}\frac{\partial x}{\partial \epsilon}\Big|_{x(0)} + \epsilon \frac{\partial f}{\partial \epsilon}\Big|_{x(0)} \tag{1.12}$$

$$0 = 0 + \epsilon f_x . x_\epsilon + \epsilon f_\epsilon \tag{1.13}$$

$$\implies \epsilon x_\epsilon = -\frac{\epsilon f_\epsilon}{f_x} \tag{1.14}$$

Here $f_a \equiv \dfrac{\partial f}{\partial a}$. The quantity $\epsilon x_\epsilon$ is the measure of 'deviation' or change in the solution because of the perturbation and $\epsilon f_\epsilon$ is the measure of deviation in the equation itself. Since we know how much does the perturbation change the equation [through the condition of apparent consistency given by (b)] we can estimate the deviation in the solution as:

$$\frac{\epsilon x_\epsilon}{x} = -\frac{1}{f_x} \cdot \frac{r}{x} \tag{1.15}$$

The LHS is the relative error in the solution while the quantity $\dfrac{r}{x}$ on the RHS is the relative error in the equation caused by the addition of the $\epsilon$ term i.e., the perturbation term. For $f_x \ll 1$ the former (i.e., the error in the solution) is large even when $\dfrac{r}{x}$ (the error in the equation) is small. Thus, for the deviation or the error in the solution to be small for small $\epsilon$, $f_x$ has to be O(1).

## 1.3 Homework assignment 1

Water flowing from a small circular hole in a container has speed $v$ which is approximately given by $v = 0.6\sqrt{2gh}$, where $g$ is the gravitational acceleration and $h$ is the height of the water above the hole. Let $A(h)$ be the area of the cross section at height $h$.

(a) Derive:

$$\frac{dh}{dt} = -0.6\frac{A(0)}{A(h)} \tag{1.16}$$

(b) Suppose that the actual shape of the container is approximated by $A(h) = h^c$, $c$ is a constant. Solve the initial value problem. Discuss the apparent consistency of the approximation.

# Chapter 2

# Transformation to dimensionless variables

In the last chapter, we pointed out that natural scientists often cast the problems that they encounter into mathematical equations. However, more often than not, such equations (algebraic or differential problems) do not have exact solutions and certain 'trickery' needs to be employed to obtain as reasonable a solution as possible. One such protocol we discussed is to assume that one (or more) of the terms in the equation e.g., a term (or several terms) that is (are) preceded by a 'small' coefficient — $\epsilon$, can be neglected and solving the approximate equation where $\epsilon = 0$. This protocol is neither unique nor perfect and we will continue to explore its different aspect throughout this course.

## 2.1 A tale of units

An equation usually comprises of a collection of variables upon which different operators e.g., addition, subtraction, differentiation act. For a natural scientists different operators need to conserve the 'units' in an equation. For instance, an equation such as

$$\frac{du}{dt} - g + a = 0; \tag{2.1}$$

makes sense to a physicist only if $u$ has the units of velocity and both $g$ and $a$ have the units of acceleration. This way of thinking is quite convenient because by looking at the magnitude of different physical quantities, one can determine how much control that quantity (and in turn the mechanism represented by it) exercises on the system. However, our protocol relies on being able to identify small coefficients which may not be a trivial task in a 'dimensional' equation for a variety of reasons e.g., the coefficient itself may be composed of different physical quantities. To circumvent such issues, equations can be transformed from the dimensional framework to a non-dimensional one by scaling.

## 2.2 Scaling

To transform an equation from a dimensional framework to a non-dimensional one, we need to divide each of the variables (both dependent and independent) in the equation by a combination of parameters that has the same unit. Non-dimensionalizing equations not only makes it easier to compare different quantities in the equation but often also reduces the number of parameters in the problem.

### 2.2.1 Example 1: Frictionless motion of a fluid parcel on the $f$-plane

The equation of friction-less motion of a fluid parcel in 'natural coordinates' (Martin, 2013, refer to page 93-97) is given by:

$$\frac{v^2}{R} - fv = fv_g \tag{2.2}$$

where $v = |\vec{v}| > 0$ is the magnitude of velocity, $f$ is the constant Coriolis frequency and $v_g$ is the geostrophic velocity associated with the constant pressure-gradient force i.e., $v_g$ satisfies the equation:

$$v_g = -\frac{1}{\rho f}\frac{\partial p}{\partial n}. \tag{2.3}$$

Here, $p$ is the pressure and $n$ is the perpendicular direction to the direction of motion i.e., directed to the left of $\vec{v}$ (so $\vec{v}$, $\hat{n}$ and $\hat{k}$ form a right-handed triplet where $\hat{n}$ is the unit vector in the normal direction, $n$, and $\hat{k}$ is the unit vector in the upward direction relative to the plane spanned by $n$ and $\vec{v}$). $R$ is the radius of curvature of parcel trajectory, $R > 0$ if $\hat{n}$ is directed toward the center of the curvature (counterclockwise flow) and $R < 0$ if $\hat{n}$ is directed away from the center (clockwise flow). Table 2.1 summarizes the dimensions of the variable, $v$, and the three parameters in the equation (2.2).

| Variable | Dimension | Parameters | Dimensions |
|---|---|---|---|
| $v$ | $\mathrm{LT}^{-1}$ | $f$ | $\mathrm{T}^{-1}$ |
| | | $v_g$ | $\mathrm{LT}^{-1}$ |
| | | $R$ | L |

Table 2.1: Dimensions of variables and parameters in (2.2)

Equations (2.2) can be non-dimensionalized by scaling $v$ on either $v_g$ or $Rf$ since both have dimensions of velocity. Let $R_0 = \dfrac{v}{Rf}$, i.e., $v = R_0(Rf)$. Substituting this in (2.2) yields:

$$\frac{R_0^2(Rf)^2}{R} + fR_0(Rf) = fv_g \tag{2.4}$$

$$\implies \underbrace{R_0^2}_{\text{dimensionless}} + \underbrace{R_0}_{\text{dimensionless}} = \underbrace{\frac{v_g}{Rf}}_{\text{dimensionless}} \tag{2.5}$$

Substituting $\dfrac{v_g}{Rf} = R_{0g}$ in (2.5) yields:

$$R_0^2 + R_0 = R_{0g} \tag{2.6}$$

which is an equation with a single non-dimensional parameter – $R_{0g}$. Figure 2.1 illustrates the solutions of (2.6) for several values of $R_{0g}$. The different regimes on the solution plane, where the abscissa is $R_0^2 + R_0$, has different dynamical meteorology regimes e.g., High vs. Low, Cyclonic vs Anti-cyclonic flows and Regular vs. Anomalous flows (see Fig. 2 in Cohen et al., 2015). Note although $R$, $f$ and $v_g$ can all assume positive and negative values, the solution of equation (2.6) covers all sign- combinations.

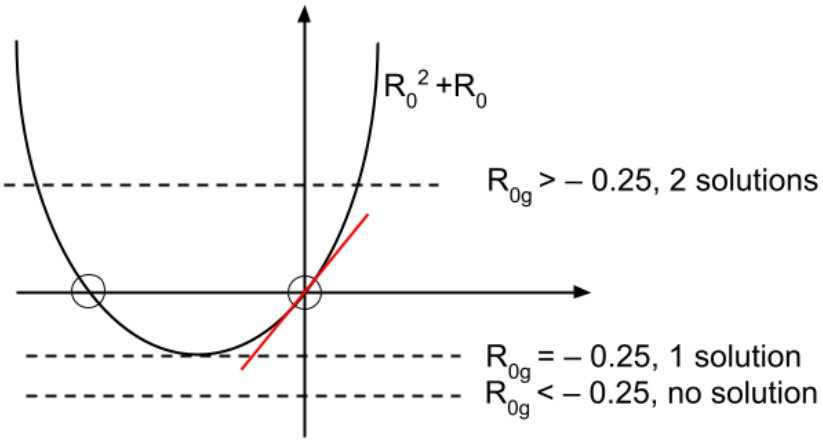


Figure 2.1: Solutions of (2.6) for different $R_{0g}$ regimes. The two circles denote the solutions $R_0 = -1$ and $R_0 = 0$ for $R_{0g} = 0$. The red line, $R_0 = R_{0g}$, is tangential to the curve $R_0^2 + R_0$ at $(0, 0)$.

An alternate way of non-dimensionalizing Equation (2.2) is to use $v_g$ to scale $v$ instead of $Rf$. This choice of scaling yields:

$$\frac{\hat{v}^2 (v_g)^2}{R} + f\hat{v}(v_g) = fv_g \tag{2.7}$$

$$\implies \frac{v_g}{fR}\hat{v}^2 + \hat{v} = 1 \tag{2.8}$$

The substitution $\frac{v_g}{fR} = R_{0g}$ then yields:

$$R_{0g}\hat{v}^2 + \hat{v} = 1 \tag{2.9}$$

For $R_{0g} \ll 1$ i.e., for $Rf \gg v_g$, (2.9) yields the geostrophic solution. It is rather non-trivial to obtain this solution from (2.6) because the parameter $R$ is used in the scaling itself. **It is advised not to use in the scaling a parameter that one wishes to vary in the interpretation**. The choice of scaling is determined by the form of the non-dimensional equation and by the analysis one wishes to perform on it. For example, setting $R_{0g} = 0$ in (2.9) yields the geostrophic solution while the same substitution yields the Inertial flow in (2.8)

As is evident from (2.5) and (2.8), there are two scales for $v$. Likewise, in any problem there are multiple combination of parameters over which the dimensional variables can be scaled, however, not every scaling yields solvable non-dimensional equations. We show this in our next example.

### 2.2.2 Example 2: The projectile problem

We explore the dynamics of a small projectile launched upwards from the ground (i.e., $z = 0$ with vertical velocity $v$. Considering the law of gravitational attraction between two bodies (that decays with the square of the distance between them) the equation of motion for such a projectile is given by:

$$\frac{d^2 z}{dt^2} = -g\frac{a^2}{(a+z)^2} \tag{2.10}$$

where $z$ is the displacement of the projectile from the surface ($z = 0$), $g$ is the gravitational acceleration at the surface and $a$ is the radius of Earth. The initial conditions associated with (2.10) are:

$$z(0) = 0 \quad \text{and} \quad \left.\frac{dz}{dt}\right|_{z=0} = v. \tag{2.11}$$

The classical approach to solving this problem is to assume that $z \ll a$ i.e., the gravitational acceleration does not vary with the height of the projectile from the surface (i.e., $a + z$ in the denominator of the RHS is replaced by the constant $a$) and solve for $z$ as a function of $t$ which yields:

$$z = vt - \frac{1}{2}gt^2 \tag{2.12}$$

The maximum height reached by the projectile (found by setting $\frac{dv}{dz} = 0$) is $z_{max} = \dfrac{v^2}{2g}$ so to guarantee that $z \ll a$ uniformly at all times, one has to require only that $z_{max} \ll a$ i.e., $v^2 \ll 2ag$. Having analyzed the $0^{th}$ order dynamics We turn now to higher order terms in the dynamics by analyzing first the ramifications of a non-dimensional formulation of this problem.

There are two variables in this problem: $z$ (the dependent variable) and $t$ (the independent variable). Equation (2.10) and the associated initial conditions, have three parameters: $a$, $g$ and $v$. The dimensions of the variables $z$ and $t$ are L and T, respectively while the dimensions of the parameters $a$, $g$ and $v$ are L, $\mathrm{LT}^{-2}$ and $\mathrm{LT}^{-1}$ respectively.

The scaling starts by selecting a combination of parameters that has the same dimension as one of the variables (dependent as well as independent) and doing so for all the variables in the problem (equations and associated initial and/or boundary conditions). It should be emphasized that the scaling is not unique and in most cases there are several ways for scaling the variables that appear in a particular problem. In the projectile problem (where $z$ and $t$ are the only variables) one such possible choice of scaling is $a$ for $z$ and $\dfrac{a}{v}$ for $t$ i.e., letting: $\hat{z} = \dfrac{z}{a}$ and $\hat{t} = \dfrac{t}{a/v}$ in (2.10) which yields:

$$\begin{aligned}
\frac{1}{(a/v)^2}\frac{d^2}{d\hat{t}^2}(a\hat{z}) &= -g\frac{a}{(a + a\hat{z})^2} \\
\frac{v^2}{a}\frac{d^2\hat{z}}{d\hat{t}^2} &= -\frac{g}{(1+\hat{z})^2} \\
\frac{d^2\hat{z}}{d\hat{t}^2} &= -\frac{ag}{v^2}\frac{1}{(1+\hat{z})^2}
\end{aligned}$$

Substituting $\dfrac{v^2}{ag} = \epsilon$ yields:

$$\epsilon\frac{d^2\hat{z}}{d\hat{t}^2} = -\frac{1}{(1+\hat{z})^2} \tag{2.13}$$

The initial conditions can be rewritten as $\hat{z}(0) = 0$ and $\left.\dfrac{d\hat{z}}{d\hat{t}}\right|_{\hat{z}=0} = 1$. Clearly, for $\epsilon = 0$, the equation is not solvable as the second order differenial operator disappears from the governing equation. We need to find the appropriate scaling which yields a solvable equation in the limit of interest. Table 2.2 provides a summary of all the non-dimensionalized versions of (2.10) and the initial conditions (2.11). Clearly, of the 6 possible choices of scaling, the non-dimensional equation is solvable for $\epsilon = 0$ only when time is scaled on $v/g$ and length is scaled over $v^2/g$.

| | | | |
|---|---|---|---|
| Time scale<br>Length scale | $a/v$<br>$a$ | $\sqrt{a/g}$<br>$a$ | $v/g$<br>$a$ |
| ODE | $\epsilon\dfrac{d^2\hat{z}}{d\hat{t}^2} = -\dfrac{1}{(1+\hat{z})^2}$ | $\dfrac{d^2\hat{z}}{d\hat{t}^2} = -\dfrac{1}{(1+\hat{z})^2}$ | $\dfrac{d^2\hat{z}}{d\hat{t}^2} = -\dfrac{\epsilon}{(1+\hat{z})^2}$ |
| ICs | $\hat{z}(0) = 0$ & $\dfrac{d\hat{z}}{d\hat{t}}\Big\vert_{\hat{z}=0} = 1$ | $\hat{z}(0) = 0$ & $\dfrac{d\hat{z}}{d\hat{t}}\Big\vert_{\hat{z}=0} = \epsilon^{1/2}$ | $\hat{z}(0) = 0$ & $\dfrac{d\hat{z}}{d\hat{t}}\Big\vert_{\hat{z}=0} = \epsilon$ |
| Time scale<br>Length scale | $a/v$<br>$v^2/g$ | $\sqrt{a/g}$<br>$v^2/g$ | $v/g$<br>$v^2/g$ |
| ODE | $\epsilon^2\dfrac{d^2\hat{z}}{d\hat{t}^2} = -\dfrac{1}{(1+\epsilon\hat{z})^2}$ | $\epsilon\dfrac{d^2\hat{z}}{d\hat{t}^2} = -\dfrac{1}{(1+\epsilon\hat{z})^2}$ | $\dfrac{d^2\hat{z}}{d\hat{t}^2} = -\dfrac{1}{(1+\epsilon\hat{z})^2}$ |
| ICs | $\hat{z}(0) = 0$ & $\dfrac{d\hat{z}}{d\hat{t}}\Big\vert_{\hat{z}=0} = \dfrac{1}{\epsilon}$ | $\hat{z}(0) = 0$ & $\dfrac{d\hat{z}}{d\hat{t}}\Big\vert_{\hat{z}=0} = \dfrac{1}{\epsilon^{1/2}}$ | $\hat{z}(0) = 0$ & $\dfrac{d\hat{z}}{d\hat{t}}\Big\vert_{\hat{z}=0} = 1$ |

Table 2.2: Summary of all non-dimensionalized versions of (2.10); $\epsilon = v^2/ag$.

## 2.3 Conclusion: Why do we scale?

1. The non-dimensionalization of a dimensional, i.e., physical problem, reduces the number of parameters in the problem. It's important to note that some of the parameters can appear in the initial or boundary condition(s) and not in the (differential) equation itself.

2. A desired outcome of the scaling is that the relative size of a term in the problem (i.e., the controlling mechanism of a system) is accurately estimated by its coefficient.

3. Instructive scaling should yield an acceptable known solution to the problem when one of the non-dimensional parameters is set equal to 0. This usually represents one of the asymptotic limits associated with the problem under study.

## 2.4 Homework assignment 2

Verify that all the ODEs listed in Table 2.2 are correct, by non-dimensionalizing the differential problem described by equation (2.10) and the associated initial conditions (2.11) using the different scales given in the table.

# Chapter 3

# Regular perturbation methods

Natural scientists often express problems in dynamics (particle motion, flow of geophysical fluids, chemical kinetics) as differential equations. However, only a handful of these equations can be solved (integrated) to obtain exact analytical solutions. To be precise the number of solvable equations can be classified into two categories:

a) linear[1], ordinary differential equations (ODEs) with constant coefficient with simple initial/boundary conditions and

b) certain types of linear, homogeneous, partial differential equations (some extensions to non-homogeneous equations also exist).

Over time, mathematicians have developed several methods which can be employed to obtain solutions to these 'unsolvable' differential equations. One category of such methods are the perturbation methods which will be discussed at length through this course. At its crux, any perturbation method/technique involves three steps, i) finding the exact solution to a simpler version of the problem, ii) assuming that the solution to the original problem is a small 'perturbation' added to the simple solution and iii) estimating the perturbation terms.

## 3.1 Regular perturbation

We illustrate the applicability of perturbation methods by solving the following algebraic equation using the regular perturbation method:

$$x^3 - (4+\epsilon)x + 2\epsilon = 0 \tag{3.1}$$

where $\epsilon = 0.001$. For $\epsilon = 0$, the solutions to (3.1) are $x = \{-2, 0, 2\}$. We assume that for $\epsilon \neq 0$, the solution to (3.1) is of the form $x = x_i + \delta$ where $x_i$ is the solution to the simpler equation i.e., the equation with $\epsilon = 0$. Since $x_i + \delta$ is a solution to (3.1):

$$\begin{aligned} (x_i+\delta)^3 - (4+\epsilon)(x_i+\delta) + 2\epsilon &= 0 \\ x_i^3 + 3x_i^2\delta + 3x_i\delta^2 + \delta^3 - (4x_i + \epsilon x_i + 4\delta + \epsilon\delta) + 2\epsilon &= 0 \end{aligned} \tag{3.2}$$

As per our assumptions, $\delta \sim \epsilon \ll 1$ and hence the higher order terms in (3.2) can be safely neglected to yield:

$$x_i^3 + 3x_i^2\delta - 4x_i - \epsilon x_i - 4\delta + 2\epsilon = 0. \tag{3.3}$$

[1] A differential equation is linear when the sum of two of its solutions is also a solution and a constant times a solution is also a solution.

Substituting $x_i = -2$ in (3.3) yields $\delta = -\frac{\epsilon}{2}$, thus one of the (approximated) roots to (3.1) is $x = -2 + \frac{0.001}{2} = 1.9995$. The other roots can be calculated similarly.

## 3.2 Homework assignment 3

1. Calculate the roots of (3.1) by neglecting $O(3)$ or higher terms in (3.2). How many values of $\delta$ do you obtain? Are all the values acceptable perturbations to the simple solution? (Hint: Remember $\delta \sim \epsilon$)

2. Substitute $x = \sum_{j=0}^{\infty} x_j \epsilon^j$, in (3.1) and find the approximate roots of the equation.

## 3.3 Power series expansion

This ad-hoc method that we employed earlier to solve (3.1) is a special case of a more general procedure described as follows:

1. Assume that the roots (solution) of the algebraic (differential) equation are of the form $x = \sum_{j=0}^{\infty} x_j \epsilon^j$ (often referred to as the power series) and substitute in into the equation.

2. Collect the terms with like power of $\epsilon$.

3. For each power of $\epsilon$, set the coefficient (constituting different combinations of $x_j$s) to 0 and find the relation between $x_j$s.

### 3.3.1 Example 1: Solving algebraic equation using power series

We try to solve an algebraic equation of the form:

$$x^2 + \epsilon x - 1 = 0 \tag{3.4}$$

using the method of power series expansion.

**Step 1:** We assume that the solution to (3.4) is of the form $x = \sum_{i=0}^{\infty} x_i \epsilon^i$ and substitute it in the equation to yield:

$$\begin{aligned} \left(\sum_{i=0}^{\infty} x_i \epsilon^i\right)^2 + \epsilon \left(\sum_{i=0}^{\infty} x_i \epsilon^i\right) - 1 = 0 \\ (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots)^2 + \epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots) - 1 = 0 \\ (x_0^2 + \epsilon^2 x_1^2 + \epsilon^4 x_2^2 + 2\epsilon x_0 x_1 + 2\epsilon^3 x_1 x_2 + 2\epsilon^2 x_0 x_2 + \dots) \\ +\epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots) - 1 = 0 \end{aligned} \tag{3.5}$$

**Step 2:** Collecting like powers of $\epsilon$ yields:

$$(x_0^2 - 1) + \epsilon(x_0 + 2x_0 x_1) + \epsilon^2(x_1^2 + x_1 + 2x_0 x_2) \cdots = 0 \tag{3.6}$$

**Step 3:** Setting the coefficients of each power of $\epsilon$ to 0 yields:

$$\begin{aligned} O(\epsilon^0) : x_0^2 - 1 &= 0 \implies x_0 = \pm 1 \\ O(\epsilon^1) : x_0 + 2x_0x_1 &= 0 \implies x_1 = -\frac{1}{2} \\ O(\epsilon^2) : x_1^2 + x_1 + 2x_0x_2 &= 0 \implies x_2 = \pm\frac{1}{8} \end{aligned}$$

The (approximate) roots of (3.4) obtained using the power series method are given by:

$$x = \begin{cases} 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + \dots \\ -1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \dots \end{cases} \tag{3.7}$$

The exact roots of (3.4) are $x_{exact} = \frac{\epsilon \pm \sqrt{\epsilon^2 + 4}}{2}$. To see how the approximate roots are related to real roots, we rewrite $x_{exact}$ as:

$$\begin{aligned} x_{exact} &= -\frac{\epsilon}{2} \pm \left(1 + \frac{\epsilon^2}{4}\right)^{1/2} \\ &\approx -\frac{\epsilon}{2} \pm \left(1 + \frac{1}{2}\frac{\epsilon^2}{4} + \dots\right) \\ &= -\frac{\epsilon}{2} \pm 1 \pm \frac{\epsilon^2}{8} + O(\epsilon^3) \dots \end{aligned} \tag{3.8}$$

Here we employed the Taylor expansion: $\sqrt{(1+a)} = 1 + \frac{a}{2} - \frac{a^2}{8} + \frac{a^3}{16} \dots)$ which is valid for $a \ll 1$. Equation (3.8) shows that the power series solution given by (3.7) is precisely equal to Taylor expansion of the exact solution.

### 3.3.2 Example 2: Solving differential equation using power series

We now try to solve the following differential equation using the power series method.

$$\frac{d^2y}{dt^2} + \epsilon y = 0; \quad y(0) = 1, y'(0) = 0 \tag{3.9}$$

The exact solution to (3.9) is $y = \cos(\sqrt{\epsilon}t)$, we now apply the power series method to obtain a solution and see how it relates to the exact solution.

**Step 1:** Assuming $y(t) = \sum_{i=0}^{\infty} y_i(t)\epsilon^i$ is a solution to (3.9), the zeroth, first and second derivatives of $y(t)$ are given by:

$$\begin{aligned} y(t) &= y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) \dots \\ y'(t) &= y_0'(t) + \epsilon y_1'(t) + \epsilon^2 y_2'(t) \dots \\ y''(t) &= y_0''(t) + \epsilon y_1''(t) + \epsilon^2 y_2''(t) \dots \end{aligned}$$

Substituting the aforementioned expansions in (3.9) yields:

$$y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' \cdots + \epsilon(y_0 + \epsilon y_1 + \epsilon^2 y_2 \dots) = 0 \tag{3.10}$$

**Step 2:** Collecting like powers of $\epsilon$:

$$y_0'' + \epsilon(y_1'' + y_0) + \epsilon^2(y_2'' + y_1) + \epsilon^3(y_3'' + y_2) + \cdots = 0 \tag{3.11}$$

**Step 3:** Setting the coefficients of each power of $\epsilon$ to 0 and satisfying the boundary conditions for every order of $\epsilon$ :

$$O(\epsilon^0) : y''(t) = 0 \implies y_0(t) = \alpha t + \beta$$

The boundary conditions dictate:

$$\begin{aligned} y_0(0) = \alpha(0) + \beta = 1 &\implies \beta = 1 \\ y_0'(0) = \alpha = 0 &\implies \alpha = 0 \end{aligned}$$

Therefore $y_0(t) = 1$. Similarly,

$$\begin{aligned} O(\epsilon^1) &: \epsilon y_1'' + y_0 = 0 \\ &\implies \epsilon y_1'' + 1 = 0 \\ &\implies y_1(t) = -\frac{t^2}{2} + \alpha t + \beta \end{aligned}$$

Applying the boundary conditions, $y_1(0) = 0$ and $y_1'(0) = 0$, yields:

$$\begin{aligned} -\frac{(0)^2}{2} + \alpha(0) + \beta = 0 &\implies \beta = 0 \\ 0 + \alpha = 0 &\implies \alpha = 0 \end{aligned}$$

Thus, $y_1(t) = -\frac{t^2}{2}$. We repeat the same exercise again for $O(\epsilon^2)$ and obtain $y_2(t) = \frac{t^4}{24}$.

The solution to (3.9) obtained by the power series method is given by:

$$y(t) = 1 - \frac{\epsilon t^2}{2} + \frac{\epsilon^2 t^4}{24} + \dots \tag{3.12}$$

which is the same as Taylor series expansion of the exact solution $y(t) = \cos(\sqrt{\epsilon}t)$ (Taylor expansion of $\cos(x)$ about $x = 0$: $1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$).

The power series solutions are not always Taylor series expansions of familiar functions like sin, cos or exp because more often than not, differential equations do not have 'nice-looking' exact solutions. In the next example we solve a problem which does not have an exact solution using the power series method and compare the approximate solution to the numerical solution.

### 3.3.3 Example 3: Solving differential equation without an exact solution

The problem at hand is given by:

$$\frac{d^2y}{dt^2} + \epsilon y^2 = 0; \quad y(0) = 1, y'(0) = 0 \tag{3.13}$$

**Step 1:** Assuming $y(t) = \sum_{i=0}^{\infty} y_i(t)\epsilon^i$ is a solution to (3.13), the zeroth, first and second derivatives of $y(t)$ are given by:

$$\begin{aligned} y(t) &= y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) \dots \\ y'(t) &= y_0'(t) + \epsilon y_1'(t) + \epsilon^2 y_2'(t) \dots \\ y''(t) &= y_0''(t) + \epsilon y_1''(t) + \epsilon^2 y_2''(t) \dots \end{aligned}$$

Substituting the aforementioned expansions in (3.13) yields:

$$\begin{aligned} y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' \cdots + \epsilon(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots)^2 = 0 \\ y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' \cdots + \epsilon(y_0^2 + \epsilon^2 y_1^2 + \epsilon^4 y_2^2 + 2\epsilon y_0 y_1 + 2\epsilon^3 y_1 y_2 + 2\epsilon^2 y_0 y_2 + \dots) = 0 \end{aligned} \tag{3.14}$$

**Step 2:** Collecting like powers of $\epsilon$:

$$y_0'' + \epsilon(y_1'' + y_0^2) + \epsilon^2(y_2'' + 2y_0y_1) + \epsilon^3(y_3'' + y_1^2 + 2y_0y_2) + \cdots = 0 \tag{3.15}$$

**Step 3:** Setting the coefficients of each power of $\epsilon$ to 0 and satisfying the relevant boundary conditions for every order of $\epsilon$ yields:

$$\begin{aligned} O(\epsilon^0) : y_0'' &= 0 \implies y_0 = 1 \\ O(\epsilon^1) : y_1'' + y_0^2 &= 0 \implies y_1 = -\frac{t^2}{2} \\ O(\epsilon^2) : y_2'' + 2y_0y_1 &= 0 \implies y_2 = \frac{t^4}{12} \\ O(\epsilon^3) : y_3'' + y_1^2 + 2y_0y_2 &= 0 \implies y_3 = -\frac{t^6}{72} \end{aligned}$$

The solution $y(t)$ to (3.13) is thus given by:

$$y(t) = 1 - \epsilon\frac{t^2}{2} + \epsilon^2\frac{t^4}{12} - \epsilon^3\frac{t^6}{72} + O(\epsilon^4). \tag{3.16}$$

Figure 3.1 shows how well the power series method approximates the solution for $\epsilon = 0.1$. Despite its elegance, the power series method can only be applied when certain conditions are satisfied. We elaborate more on this issue in the subsequent section.

## 3.4 Limitation of power series method

Consider the following algebraic equation:

$$\epsilon x^2 + x - 1 = 0 \tag{3.17}$$

A valid power series solution cannot be obtained for (3.17), because for $\epsilon = 0$ the equation has only one root. The exact solution of (3.17) is given by:

$$x_{\pm} = \frac{-1 \pm \sqrt{1 + 4\epsilon}}{2\epsilon} \approx \frac{1 \pm (1 + 2\epsilon)}{2\epsilon} \tag{3.18}$$

For $\epsilon = 0$, one of the solutions is singular (this reflects the fact that for $\epsilon = 0$ the equation becomes $x - 1 = 0$ which has one solution only) and $x_- = \frac{1}{\epsilon} - 1$ has no expansion near $\epsilon = 0$. Thus, power series expansions are valid only for the solutions that prevail when $\epsilon = 0$. More sophisticated methods to tackle equations similar to (3.17) will be discussed in the section (5).

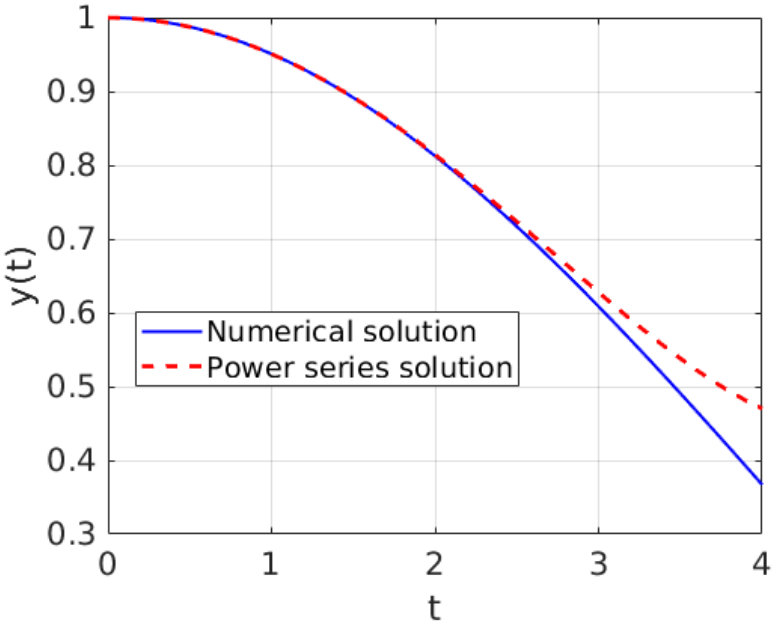


Figure 3.1: Numerical integration v/s power series method to solve (3.13) for $\epsilon = 0.1$

## 3.5 Homework assignment 4

Solve the following equation by applying the power series method

$$y' - y - \epsilon y t = 0; \quad y(0) = 1. \tag{3.19}$$

## 3.6 Bonus section: Solving inhomogeneous ODEs

An typical inhomogeneous ODE of second order is given:

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = f(t) \tag{3.20}$$

Let the two independent solutions to (3.20) for $f(t) = 0$ be $y_1(t)$, $y_2(t)$ and the general solution for $f(t) \neq 0$ be $y(t) = \alpha y_1(t) + \beta y_2(t) + y_p(t)$ where $y_p(t)$ is any particular solution to (3.20). In this section, we outline a recepie to find $y_p(t)$

Let's assume that $y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t)$, therefore:

$$y_p' = c_1'y_1 + c_1y_1' + c_2'y_2 + c_2y_2'.$$

Here we impose the following requirement:

$$c_1'y_1 + c_2'y_2 = 0 \tag{3.21}$$

Following the requirement given by (3.21), the derivative of $y_p(t)'$ (i.e., the double derivative of $y_p$) is given by:

$$y_p'' = c_1'y_1' + c_1 + y_1'' + c_2'y_2' + c_2y_2'' \tag{3.22}$$

Since, $y_p$ is a solution to (3.20), we can write:

$$y_p'' + ay_p' + by_p = f(t)$$

Substituting, $y_p''$, $y_p'$ and $y_p$ in the aforementioned equation yields:

$$\begin{aligned} c_1'y_1' + c_1 + y_1'' + c_2'y_2' + c_2y_2'' + a(c_1y_1' + c_2'y_2) + b(c_1y_1 + c_2y_2) &= f(t) \\ \implies c_1 \underbrace{(y_1'' + ay_1 + by_1)}_{0} + c_2 \underbrace{(y_2'' + ay_2 + by_2)}_{0} + c_1'y_1 + c_2'y_2' &= f(t) \\ \implies c_1'y_1' + c_2'y_2' &= f(t) \end{aligned} \tag{3.23}$$

Equation (3.23) is the second requirement, the two requirements can be cast as a single matrix equation:

$$\begin{aligned} \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} &= \begin{pmatrix} 0 \\ f \end{pmatrix} \\ \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} &= \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ f \end{pmatrix} \end{aligned} \tag{3.24}$$

The matrix equation, (3.24), can be solved to obtain $c_1'$ and $c_2'$ which can further be integrated to obtain the unknown time dependent coefficients $c_1(t)$ and $c_2(t)$. This is one of many recipes to 'guess' a particular solution to a second order ODE for which the general solutions to the homogeneous part are known.

## 3.7 Homework assignment 5

In the non-dimensional framework, the equation of motion of a projectile launched vertically from ground can be written as:

$$z''(1 + \epsilon z)^2 + 1 = 0; \quad z(0) = 0, z'(0) = 1 \tag{3.25}$$

where $\epsilon \ll 1$.

Solve the above equation using the power series method to $2^{nd}$ order and for different values of $\epsilon$, compare the power series solution with the numerical solution.

## 3.8 Asymptotic approximation

The key difference between the power series and asymptotic expansions can be bought out using the following expression:

$$y(x) = \sum_{i=0}^{N} y_i(x)\epsilon^i. \tag{3.26}$$

When (3.26) is referred to as a power series, $N \to \infty$ in while $\epsilon$ is fixed whereas when it is referred to an in asymptotic expansion, $N$ is fixed while $\epsilon \to 0$.

# Chapter 4

# Conservation laws and how to use them

Up until this point, we have discussed in detail what the power series solutions to a differential problem are and how can we use power series solutions to 'solve' differential equations which do not have nice-looking regular solutions. However, power series solutions are often tedious to calculate even for relatively simple differential equations. A work around this problem is to switch to a more 'geometric' way of thinking. The work done by Poincaré in late 1800s paved the foundation for this approach to analysing differential equations. Asymptotic approximations play a crucial role while employing this geometric approach and we will discuss it at length in the subsequent sections. Readers interested in a historical overview and a (more) elaborate perspective on the geometric approach to tackle differential equations should refer to Part I. and II. of Strogatz (2000).

## 4.1 Phase space diagrams

Any higher order differential equation can be cast into set of ODEs which is often refereed to as a system of differential equations. The general form of this system is given by:

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2, \ldots, y_N) \\ \dot{y}_2 &= f_2(y_1, y_2, \ldots, y_N) \\ &\vdots \\ \dot{y}_N &= f_N(y_1, y_2, \ldots, y_N) \end{aligned}$$

The typical approach to solving differential equations is to obtain the time dependent solution to this system. The trajectories exhibited by these time-dependent solutions are often tedious to obtain, moreover such solutions can often mask some crucial physical properties of the (differential) system like periodicity and steady state behavior.

The geometric approach dictates that the differential equations be interpreted as a space made up of $x_i$s and $\dot{x}_i$s commonly known as the phase space. Each point in the phase space represents a 'state' of the system i.e., at some given time $t$, any point on the phase space trajectory of the system will represent a unique set of values $(x_i, \dot{x}_i)$ that the system takes. For instance the phase space of the system:

$$\dot{y} = \sin(y) \tag{4.1}$$

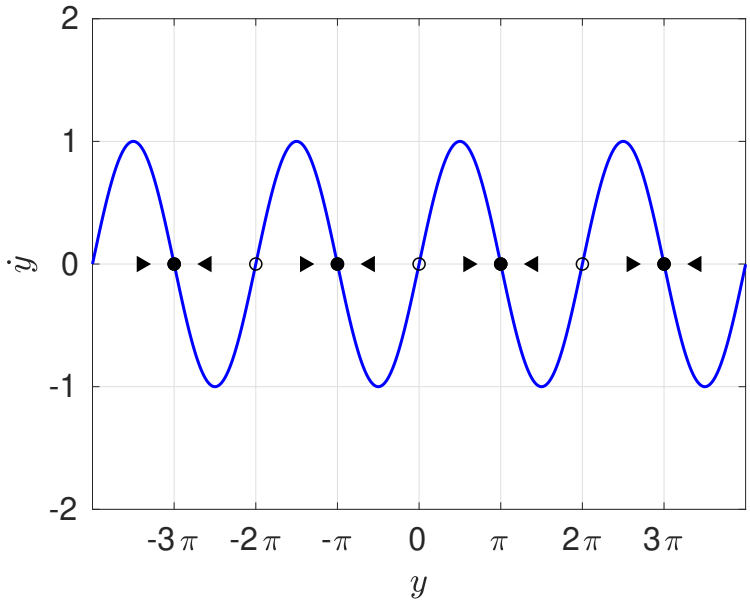


Figure 4.1: Phase space diagram corresponding to (4.1). The solid dots represent stable fixed points and the hollow circles represent unstable fixed points

is depicted in Figure 4.1.

Each point on the trajectory represents the possible combinations of $(y, \dot{y})$ that the system can take. As the system evolves in time i.e., as the value of $y$ changes, the corresponding value $\dot{y}$ necessarily lies on this trajectory. Initial conditions of any system represents a point on this trajectory.

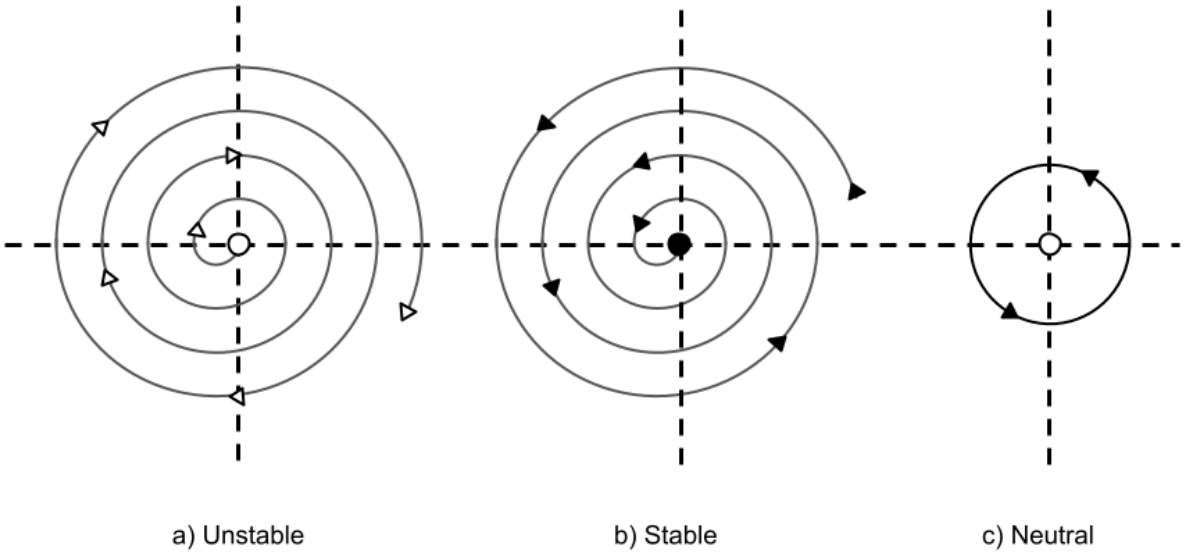


Figure 4.2: Phase space diagrams corresponding to a) unstable b) stable and c) neutral fixed point.

For certain values of $y$ on this phase space trajectory, $\dot{y} = 0$; these points are called the fixed points. Fixed points can be classified as stable or unstable. Whether a fixed point is stable or unstable is determined by how the trajectory behaves immediately prior to and

after the fixed point. For $\dot{y} > 0$, the 'flow' along the trajectory is $+ve$ and if $\dot{y} < 0$ the flow along the trajectory is $-ve$. In Figure 4.1, the black arrows along the ordinate show the direction of this flow. The trajectory flows into (out of) the stable (unstable) fixed points. Stable (Unstable) fixed points are marked by filled (hollow) circles. Stable fixed points are often referred to as the *attractors* or sinks of the system, conversely, unstable fixed points are referred to as *repellers* or sources. It is possible for a system not to flow through a fixed point but rather revolve around it in the phase space, these points are referred to as neutral fixed points. The nature of a fixed point can also be determined by evaluating the temporal evolution of a system. If the system evolves towards a fixed point in phase space with time, the fixed point is referrred to as a stable fixed point. Conversely,if a system evolves away from a fixed point with time, the point is unstable. Figure 4.2 depicts one: a) unstable, b) stable and c) neutral fixed point. The arrows indicate the direction in which the system evolves with time.

Aside from stable, unstable and neutral ones there are other kinds of fixed points as well; these will be discussed later. A fixed point can also be both stable and unstable. As an exercise, the reader is encouraged to plot the phase space diagram corresponding to $\dot{y} = y^2$. What is the fixed point for this trajectory? What kind of fixed points is it – stable/unstable/neutral?

There are other fixed points beside stable and unstable which will be discussed in the due course of time. It is important to note that two trajectories in phase space can never cross one another because at the point where they cross, the system will have two possible directions of $(\dot{y}, y)$ along which it can flow i.e., non-unique solutions. Fixed points are of interest because there are only a certain number of ways in which the trajectory can behave near a fixed point. We try to understand this with phase trajectories of a simple pendulum.

## 4.2 Simple pendulum in phase space

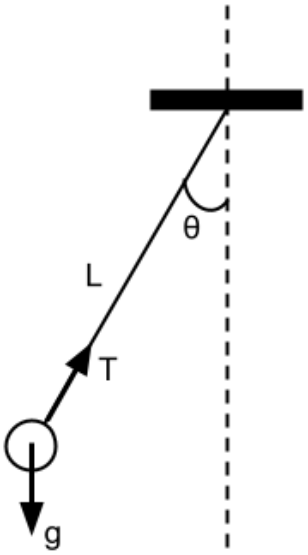


Figure 4.3: Schematic of a simple pendulum

Figure 4.3 depicts schematic of a simple pendulum. The co-sinusoidal components

of weight is balanced by the tension in the string i.e., $T = mg\cos\theta$ and the sinusoidal component of gravity accelerates the pendulum along the arc:

$$\ddot{\theta} = -\frac{g}{L}\sin\theta. \tag{4.2}$$

For small angles, $\theta \ll 1$, $\sin\theta \sim \theta$. Substituting $\frac{g}{L} = \omega^2$ and assuming that the angle of deviation is small yields:

$$\ddot{\theta} = -\omega^2\theta \tag{4.3}$$

The phase space trajectory of a simple pendulum will be a curve on the $(\dot{\theta}, \theta)$ plane. Let's try to manipulate equation (4.3) to obtain a curve on this plane.

$$\begin{aligned}
&\ddot{\theta} + \omega^2\theta = 0 \\
\implies\ &2\dot{\theta}\times\ddot{\theta} + 2\dot{\theta}\times\omega^2\theta = 0 \\
\implies\ &\frac{d}{dt}(\dot{\theta}^2) + \omega^2\frac{d}{dt}(\theta^2) = 0 \\
\implies\ &\frac{d}{dt}(\dot{\theta}^2) + \frac{d}{dt}(\omega^2\theta^2) = 0 \\
\implies\ &\dot{\theta}^2 + \omega^2\theta^2 = E
\end{aligned} \tag{4.4}$$

For $\omega^2 = 1$, the expression given by (4.4) is the equation of a circle (of an ellipse for $\omega^2 \neq 1$) and $E$ is the constant of integration. The phase space trajectories of a simple pendulum for different values of initial conditions (represented by different energies $E$) are depicted in the figure below.

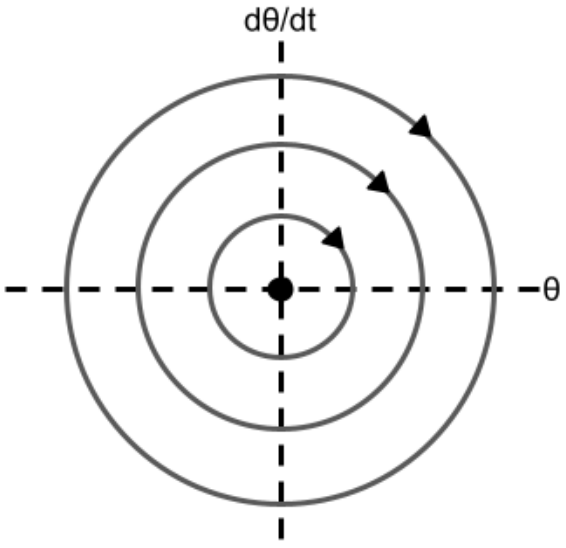


Figure 4.4: Phase space trajectory of a simple pendulum for different initial conditions (energy).

Devising an analytic expression for the trajectory of a system in phase-space is a often not trivial, however, a lot of insight can be drawn from the behavior of the system near critical points. The recipe to do so is as follows:

1. Rewrite the higher order differential equation(s) describing the system as a set of multiple first order differential equations.

2. Identify the critical points of the set of ODEs.

3. Linearize the RHS of each ODE and find the eigenvalues of the system.

We go back to the equation of motion of a simple pendulum, (4.2), and set $\frac{g}{L} = \omega^2 = 1$ to obtain:

$$\ddot{\theta} = -\sin\theta.$$

**1:** This second order differential equation can be re-written as the following set of ODEs:

$$\dot{\theta} = y \tag{4.5}$$
$$\dot{y} = -\sin\theta \tag{4.6}$$

**2:** There are multiple critical points for this system: $(y = 0,\ \theta = n\pi)$ ,where $n$ is any integer.

**3:** Linearizing the sytem of ODEs, say around the critical point: $(y = 0, \theta = \pi)$ yields:

$$(\pi \dot{+} \delta) = \dot{\delta} = y \tag{4.7}$$
$$\dot{y} = -\sin(\pi + \delta) = -(\sin\pi\cos\delta + \cos\pi\sin\delta) \approx \delta \tag{4.8}$$

which, in the matrix formalism of differential equations can be written as:

$$\begin{pmatrix} \dot{\delta} \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{M} \begin{pmatrix} \delta \\ y \end{pmatrix} \tag{4.9}$$

and the eigenvalues of M can be calculated by solving the equation: $det(M - \lambda I) = 0$, where *det* is the determinant, $\lambda$ is the eigenvalue and $I$ is identity matrix. The eigen values obtained are: $\lambda = \pm 1$. The curve in the phase space will behave like the function $e^{\pm 1}$ near the critical point $(\theta, \dot{\theta}) = (\pi, 0)$. This point where the system shows growth and decay simultaneously is known as a saddle point.

Figure 4.5 shows the trajectories of the simple pendulum in phase space for different set of initial conditions (representing different energies).

## 4.3 Damped pendulum

The equation of motion for a simple pendulum, given by (4.2), can be modified to add a damping term:

$$\ddot{\theta} + \gamma\dot{\theta} + \omega^2\sin\theta = 0 \tag{4.10}$$

**Step 1:** For $\omega^2 = 1$, (4.10) can be re-written as the following set of ODEs, by substituting $\dot{\theta} = y$:

$$\dot{\theta} = y \tag{4.11}$$
$$\dot{y} = -\sin\theta - \gamma y \tag{4.12}$$

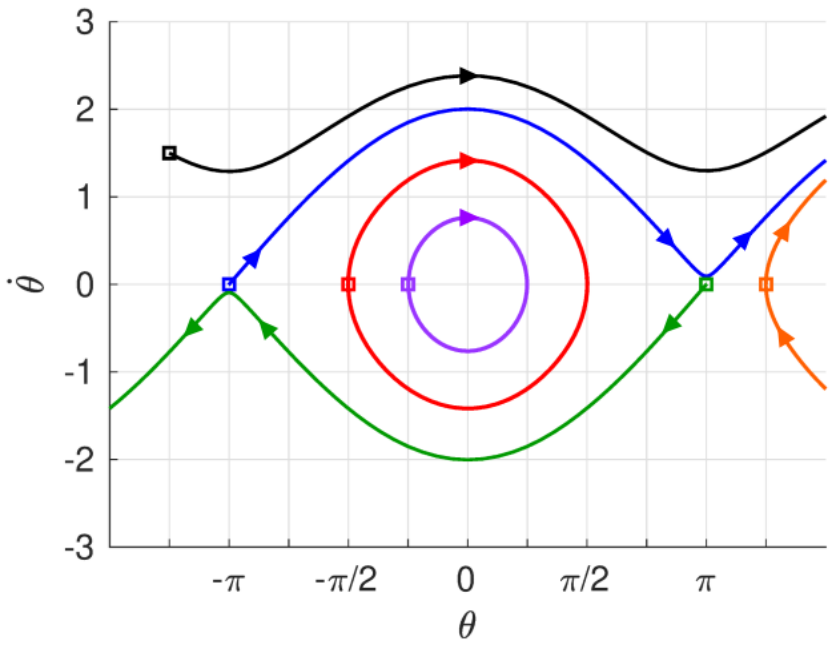


Figure 4.5: Phase space plot for a pendulum where $\omega^2 = 1$. For each trajectory, squares represent the initial condition and the arrows show the direction in which the system evolves in the phase space with time.

**Step 2:** The critical points for this system is $[y = 0,\ \theta = \sin^{-1}(-\gamma y) \approx -\gamma y - \dfrac{(\gamma y)^3}{6}]$.

**Step 3:** Linearizing the system of ODEs, say around the critical point yields:

$$\dot{\delta} = y \tag{4.13}$$
$$\dot{y} = -\delta - \gamma y \tag{4.14}$$

which is written in matrix formulation as:

$$\begin{pmatrix} \dot{\delta} \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix}}_{M} \begin{pmatrix} \delta \\ y \end{pmatrix}. \tag{4.15}$$

The eigenvalues of M, calculated by solving the equation $|M - \lambda I| = 0$, are: $\lambda = -\dfrac{\gamma}{2} \pm i\sqrt{1 - \dfrac{\gamma^2}{4}}$. For $\gamma \ll 1$, the eigenvalues reduce to $\lambda \approx -\dfrac{\gamma}{2} \pm i$. In this limit the system has two decaying and counter-rotating solutions (in the $+\theta$ and $-\theta$ directions): $\exp\left(-\dfrac{\gamma}{2}t - it\right)$ and $\exp\left(-\dfrac{\gamma}{2}t + it\right)$

## 4.4 Motion of a bead on a rotating planet

In the previous sections, we examined the dynamics of simple pendulum by identifying critical points in the equations of motion and drawing the phase space portrait of the system based on the behavior of the eigenvalues of the system near these fixed point. We also identified that the energy (potential+kinetic) is a conserved quantity of the system

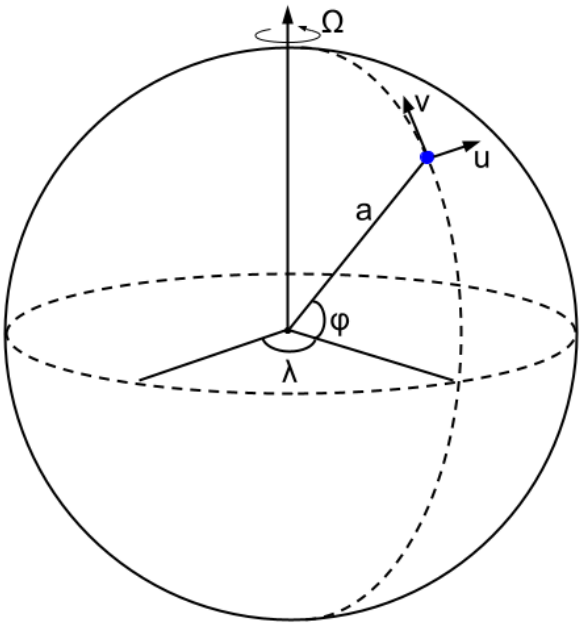


Figure 4.6: Schematic of a bead moving on a friction-less planet

and it determines what the trajectory of the system will be in the phase space (see Figure 4.5).

We now try to do the same for a more complex system — equations of motion of a bead on a rotating planet. The equations of motion for this bead is given by:

$$\frac{du}{dt} = v \sin\phi \left(2\Omega - \frac{u}{a\cos\phi}\right) \tag{4.16}$$

$$\frac{dv}{dt} = -u \sin\phi \left(2\Omega + \frac{u}{a\cos\phi}\right) \tag{4.17}$$

$$\frac{d\phi}{dt} = \frac{v}{a} \tag{4.18}$$

$$\frac{d\lambda}{dt} = \frac{u}{a\cos\phi}. \tag{4.19}$$

Here $\phi, \lambda$ are the latitudinal and longitudinal coordinates of the bead moving on a planet of radius $r$ and rotation frequency $\Omega$, and $u, v$ are the zonal and meridional velocity components of the said bead. We non-dimensionalize the equations of motion by scaling time over $\dfrac{1}{2\Omega}$ and $u, v$ over $2\Omega a$ to obtain the following non-dimensional equations:

$$\frac{du}{dt} = v \sin\phi \left(1 + \frac{u}{\cos\phi}\right) \tag{4.20}$$

$$\frac{dv}{dt} = -u \sin\phi \left(1 + \frac{u}{\cos\phi}\right) \tag{4.21}$$

$$\frac{d\phi}{dt} = v \tag{4.22}$$

$$\frac{d\lambda}{dt} = \frac{u}{\cos\phi}. \tag{4.23}$$

Our aim is to identify the conserved quantities in this system of equations. Since $\lambda$ does not apper on the RHS of any equation, we can identify that equation (4.23) does not affect the solution of system and should not be used to find a conserved quantity. We also note, some structural similarity between (4.20) and (4.21). Multiplying (4.20) with $u$, (4.21) with $v$ and adding the two yields:

$$\begin{aligned} u.\frac{du}{dt} + v.\frac{dv}{dt} &= 0 \\ \implies \frac{d}{dt}\left[\frac{1}{2}(u^2+v^2)\right] &= 0 \\ \implies \frac{u^2}{2} + \frac{v^2}{2} &= E \end{aligned} \tag{4.24}$$

here $E$ is the energy of the bead in motion. To come up with another conserved quantity we try to eliminate the time dependence using the division operation and the remaining combinations of equations. Consider two possibilities: (i) dividing (4.20) by (4.22) and (ii) dividing (4.21) by (4.22). Alternative (ii) is not a viable approach because the resulting equation has three unknowns. On the other hand, (i) yields:

$$\begin{aligned} \frac{du}{d\phi} &= \sin\phi\left(1+\frac{u}{\cos\phi}\right) \\ \frac{du}{\sin\phi\ d\phi} &= \left(1+\frac{u}{\cos\phi}\right) \\ \frac{du}{-(\cos\phi)'} &= \left(1+\frac{u}{\cos\phi}\right), \end{aligned}$$

substituting $\cos\phi$ with $s$ yields:

$$\begin{aligned} -\frac{du}{ds} &= \left(1+\frac{u}{s}\right) \\ \frac{du}{ds} + \frac{u}{s} &= -1 \end{aligned} \tag{4.25}$$

To solve (4.25), we first find a solution to its homogeneous part.

$$\begin{aligned} \frac{du}{ds} + \frac{u}{s} &= 0 \\ \frac{du}{u} &= -\frac{ds}{s} \\ \ln u &= \ln D - \ln s \\ u &= \frac{D}{s} \end{aligned}$$

where $D$ is a constant. A particular solution to (4.25) is $u = -\frac{s}{2}$, the general solution then becomes:

$$u = \frac{D}{s} - \frac{s}{2}$$

substituting $s = \cos\phi$ and simplifying further yields:

$$\begin{aligned} u &= \frac{2D - \cos^2\phi}{2\cos\phi} \\ \implies 2u\cos\phi + \cos^2\phi &= 2D \\ \implies \cos\phi\left(\frac{\cos\phi}{2} + u\right) &= D. \end{aligned} \tag{4.26}$$

Equations (4.24), and (4.26) are the two conserved quantities corresponding to the system of equations under consideration. It is fairly obvious to see that (4.24) is the non-dimenionalized version of energy conservation. In the spirit of ascribing physical meaning to an equation, we ask the question: "what does the quantity given by LHS of (4.26) represent?"

For any (classical) moving particle, the two obvious conserved quantities which come to a natural scientist's mind are energy and momentum. However, for any particle subject to rotational motion about an axis, angular momentum is the second conserved quantity. Angular momentum is defined as:

$$\vec{\mu} = \vec{r} \times \vec{v} \tag{4.27}$$

Any point at latitude $\phi$ on a planet is moving eastward with the dimensional velocity $u_e = \Omega a\cos\phi$. Since we scale the speed on $2\Omega a$, the non-dimensional eastward velocity of the point is $u_e = \dfrac{\cos\phi}{2}$. The total eastward velocity component of the bead at that point is thus $(u_e + u)$. We now need the component of the radius vector $(\vec{a})$ perpendicular to the axis of rotation, which is $a\cos\phi$ in dimensional units or simply $\cos\phi$ in non-dimensional ones. Multiplying $(u_e + u)$ with $\cos\phi$ yields $\cos\phi\left(\dfrac{\cos\phi}{2} + u\right)$: the angular momentum of a bead moving on a rotating planet, which is the second conserved quantity corresponding to the system of equations under consideration.

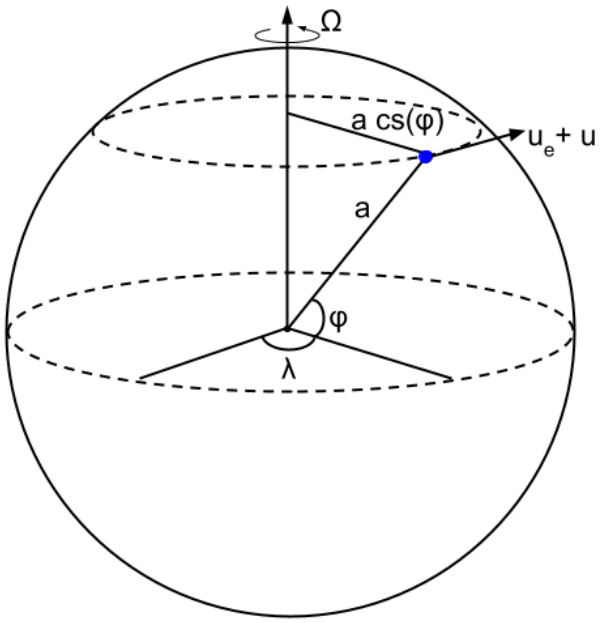


Figure 4.7: Illustration of the eastward velocity of a bead along the latitudinal circle of radius $a\cos(\phi)$

Identifying conserved quantities not only reduces the number of equations describing any physical system but also helps identify certain dynamical aspects of the said system which are otherwise difficult to unveil. We further elaborate on this by considering the motion of the bead which starts moving at $t = 0$ from longitude $\lambda(t = 0) = 0$ with some eastward velocity $u(t = 0) > 0$ and no northward velocity $v(t = 0) = 0$ The bead's initial latitude $\phi$ can vary from 0 to $\frac{\pi}{2}$.

If the the bead is at the Equator ($\phi = 0$), it has to necessarily move eastward because if it moves northward $\cos\phi$ and in turn $u$ decreases, which violates the conservation of $D$. Figure 4.8 and 4.9 and shows the trajectory of the bead when it is launched with an initial ($u_0$,$v_0$) = (0.1,0) from different latitudes.

Figure 4.8(a)-(d) and 4.9(e) illustrate that the bead can cross the Equator for different values of $u$ and $\phi$. We now determine the latitude at which if the bead has an initial westward velocity of $u_0$, it will never cross the Equator and will perpetually move eastward along the Equator with the same speed ($u_0$). For this exercise, the initial and final $D$ of the bead are given by $\phi_0$ is $D(t = 0) = \cos\phi_0 \left(\frac{\cos\phi_0}{2} + u_0\right)$ and $D(t \to \infty) = \left(\frac{1}{2} - u_0\right)$ respectively. Equating the two and solving for $\phi_0$ yields:

$$\begin{aligned} \cos\phi_0 \left(\frac{\cos\phi_0}{2} + u_0\right) &= \left(\frac{1}{2} - u_0\right) \\ \implies u_0(1 + \cos\phi_0) &= \frac{1}{2}\left(1 - \cos^2\phi_0\right) \\ \phi_0 &= \cos^{-1}\left(1 - 2u_0\right). \end{aligned} \tag{4.28}$$

Corollary, a bead with an initial westward velocity of $u_0 = \frac{1 - \cos\phi_0}{2}$, where $\phi_0$ is the latitude, will not cross the Equator and travel along it perpetually in the eastward direction.

In the last part of this chapter, we will determine the fixed points of the system of equations (4.20) - (4.23). To do so we write $u$ as $u = \frac{D}{\cos\phi} - \frac{\cos\phi}{2}$ and substitute it in (4.23), which yields:

$$\begin{aligned} \frac{d\lambda}{dt} &= \frac{1}{\cos\phi}\left(\frac{D}{\cos\phi} - \frac{\cos\phi}{2}\right) \\ \frac{d\lambda}{dt} &= \left(\frac{D}{\cos^2\phi} - \frac{1}{2}\right) \end{aligned} \tag{4.29}$$

Substituting the expression for $u$ in (4.21) and simplifying yields:

$$\begin{aligned} \frac{dv}{dt} &= -\left(\frac{D}{\cos\phi} - \frac{\cos\phi}{2}\right)\sin\phi\left(1 + \frac{u}{\cos\phi}\right) \\ \frac{dv}{dt} &= \frac{\sin 2\phi}{2}\left(\frac{1}{4} - \frac{D^2}{\cos^4\phi}\right) \end{aligned} \tag{4.30}$$

Equations (4.29) and (4.30) along with

$$\frac{d\phi}{dt} = v \tag{4.31}$$

$$\frac{dD}{dt} = 0 \tag{4.32}$$

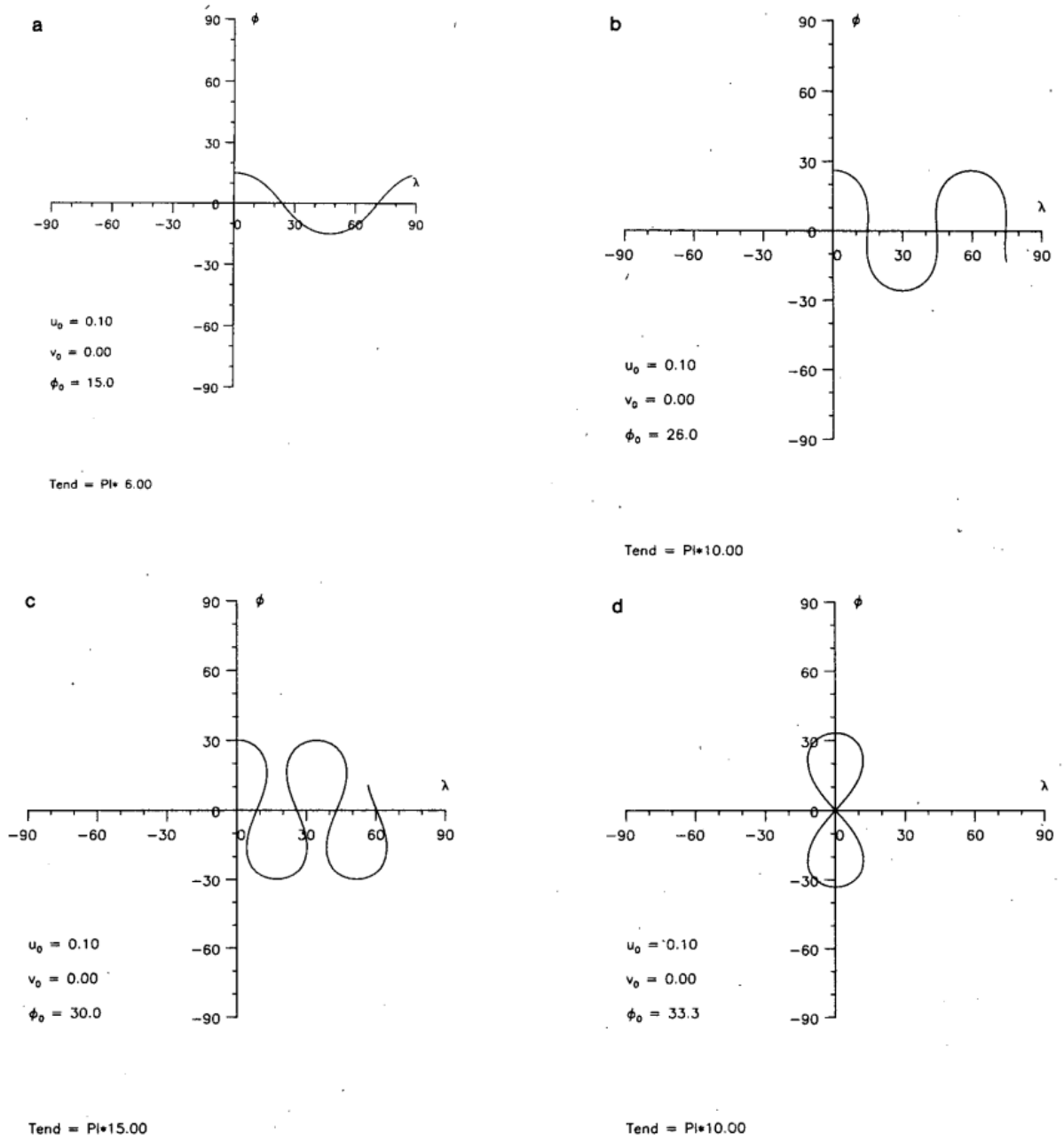


Figure 4.8: Trajectories of a bead starting with an initial Eastward velocity with increasing latitudes. [Borrowed from Paldor and Killworth (1988) with consent from the corresponding author]

are a second set of equations that describe the motion of bead on a rotating planet. In this system, however, one of the conserved quantities ($D$) is a part of the system and not an independent quantity formulated from the dependent variables in the system [as is the case of equations (4.20) - (4.23)]. In the new framework, the second conserved quantity, $E$, is given by:

$$E = \frac{v^2}{2} + \frac{1}{2}\left(\frac{D}{\cos\phi} - \frac{\cos\phi}{2}\right)^2 \tag{4.33}$$

.

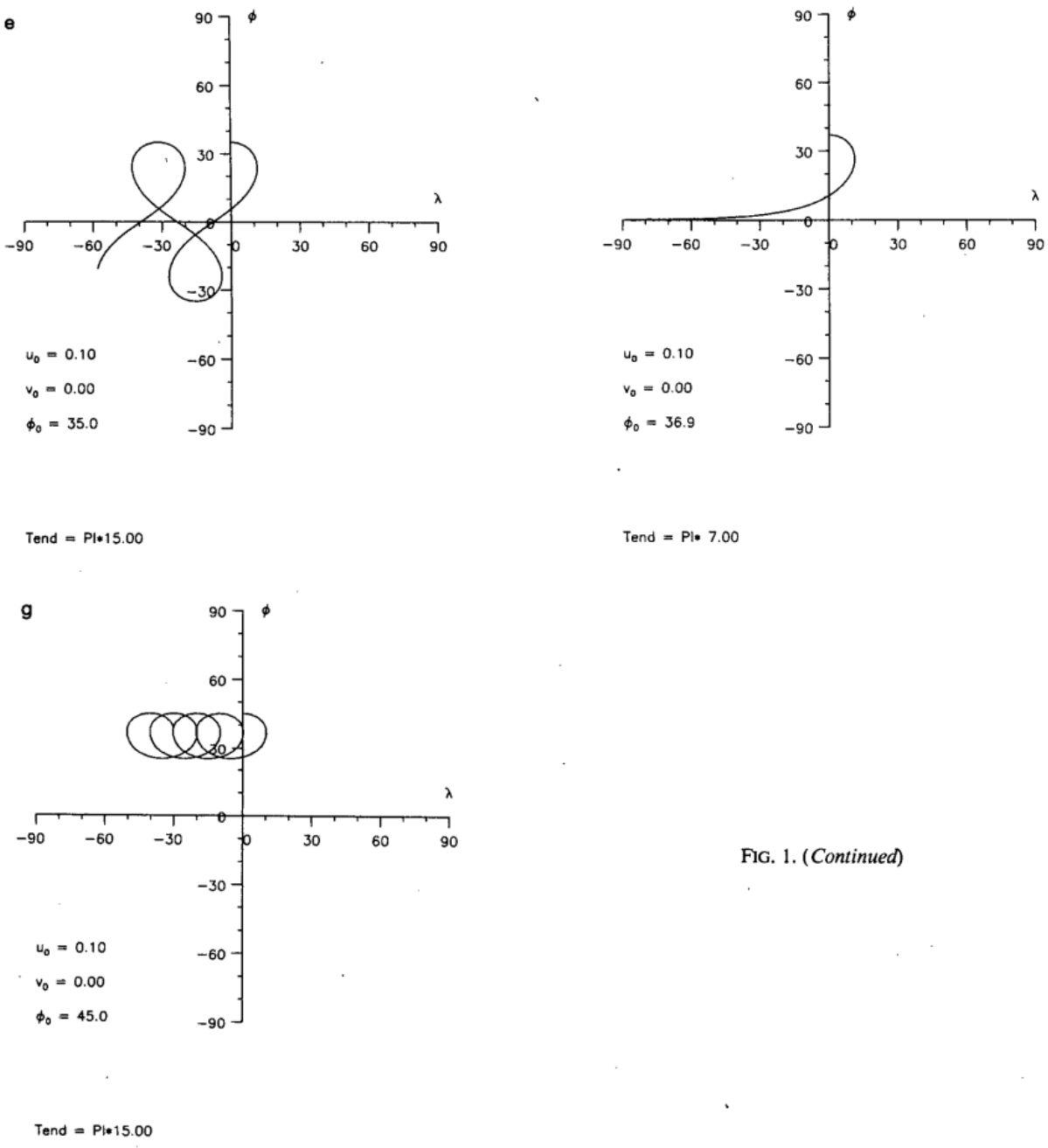


FIG. 1. (*Continued*)

Figure 4.9: Figure 4.8 continued

## 4.5 Homework assignment 6

Derive the system of equations which describe the motion of a bead on the Equatorial $\beta$ plane i.e., $\cos\phi = 1$ and $\sin\phi \approx \phi$. Also find the expression for the conserved quantities.

## 4.6 Phase space behavior of the bead on a rotating planet

In the next part of this chapter, we are going to:

- identify the fixed points corresponding to the system of equations (4.29)-(4.32) and

- determine the behavior of the system near the fixed points.

In a steady state, $\frac{dv}{dt} = 0$, which is true when $v = 0$ (trivial) or when $\frac{\sin 2\phi}{2}\left(\frac{1}{4} - \frac{D^2}{\cos^4\phi}\right) = 0$. Examining the later expression further yields two possibilities:

$$\begin{aligned} \sin 2\phi &= 0 \\ \phi_s &= n\pi \end{aligned} \tag{4.34}$$

or

$$\begin{aligned} \frac{1}{4} - \frac{D^2}{\cos^4\phi} &= 0 \\ \phi_s &= \cos^{-1}(\pm\sqrt{2D}) \end{aligned} \tag{4.35}$$

The solution $\phi_s = n\pi$ has only one valid solution $\phi_s = 0$ i.e., the Equator because $\phi$ varies between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$. The solution $\phi_s = \cos^{-1}(\pm\sqrt{2D})$ is only valid for $D < \frac{1}{2}$. Near $\phi = \phi_s$ the function $F(\phi) = \frac{\sin 2\phi}{2}\left(\frac{1}{4} - \frac{D^2}{\cos^4\phi}\right)$ is expanded to first order by:

$$F(\phi) = \underbrace{F(\phi_s)}_{0} + \frac{\partial F}{\partial \phi}\Big|_{\phi=\phi_s}(\phi - \phi_s) + \dots. \tag{4.36}$$

Evaluating $\frac{\partial F}{\partial \phi}$ yields:

$$\begin{aligned} \frac{\partial F}{\partial \phi} &= \frac{\partial}{\partial \phi}\left(\frac{\sin 2\phi}{2}\right).\left[\frac{1}{4} - \frac{D^2}{\cos^4\phi}\right] + \frac{\sin 2\phi}{2}.\frac{\partial}{\partial \phi}\left[\frac{1}{4} - \frac{D^2}{\cos^4\phi}\right] \\ &= \cos 2\phi\left[\frac{1}{4} - \frac{D^2}{\cos^4\phi}\right] + \frac{\sin 2\phi}{2}.\left[-4D^2\tan\phi\sec^4\phi\right] \\ &= \cos 2\phi\left[\frac{1}{4} - \frac{D^2}{\cos^4\phi}\right] - \sin^2\phi\frac{4D^2}{\cos^4\phi}. \end{aligned} \tag{4.37}$$

Thus,

$$\frac{\partial F}{\partial \phi}\Big|_{\phi_s=0} = \frac{1}{4} - D^2. \tag{4.38}$$

Substituting from (4.38) in (4.36) yields:

$$F(\phi \to \phi_s) = \left(\frac{1}{4} - D^2\right)(\phi - \phi_s) \tag{4.39}$$

For $D^2 > \frac{1}{4}$, the behavior of the solution is elliptical and for $D^2 < \frac{1}{4}$, the behavior is hyperbolic and $D = \frac{1}{2}$ is the bifurcation point.

We now discuss the behavior of the solution for the second case i.e., $\dfrac{D^2}{\cos^4\phi_s} = \dfrac{1}{4}$. The derivative $\dfrac{\partial F}{\partial \phi}$ given by (4.6) simplifies to:

$$\begin{aligned}
\frac{\partial F}{\partial \phi}\Big|_{\phi_s \neq 0} &= \cos 2\phi \underbrace{\left[\frac{1}{4} - \frac{D^2}{\cos^4\phi_s}\right]}_{0} - \sin^2\phi_s \underbrace{\frac{4D^2}{\cos^4\phi_s}}_{1} \\
\frac{\partial F}{\partial \phi}\Big|_{\phi_s \neq 0} &= -\sin^2\phi_s
\end{aligned} \tag{4.40}$$

# Chapter 5

# The WKB method

As mentioned briefly in subsection 3.4 the straightforward approach of the simple, regular perturbation, methods cannot always capture the complicated behavior of a solution near a singular point of a differential equation. There are two main reasons for the poor representation of the dynamical behavior of the solution near a singular point: **dispersion** and **dissipation**. To understand the subtleties associated with these two extreme behaviors, it is instructive to examine the solutions of the linear, $2^{nd}$ order, ODE $\epsilon\frac{d^2y}{dt^2} - y = 0$ in the $\epsilon \to 0$ limit. The two solutions of this equation are: $y_\pm e^{\pm t/\sqrt{\epsilon}}$ where $y_\pm$ are the two amplitudes. When $\epsilon \to 0_+$ (i.e., $\epsilon$ approaches 0 through a set of positive numbers), the solution $y_- e^{-t/\sqrt{\epsilon}}$ vanishes rapidly (dissipates) when $t$ increases from 0 i.e., it has a local breakdown near the $t = 0$ point. On the other hand when $\epsilon \to 0_-$ (i.e., $\epsilon$ approaches 0 through a set of negative numbers) $y(t) \propto \sin(t/\sqrt{\epsilon})$ undergoes "infinitely" many oscillations over the [0, 1] $t$-interval. This is called dispersion and the breakdown of $y(t)$ in this case occurs over an $O(1)$ interval of $t$ i.e., the breakdown of $y(t)$ is global instead of local.

The WKB (named after Wentzel, Kramers and Brillouin; sometimes referred to as WKBJ to include Jeffreys) method is designed to handle both local and global breakdowns of solutions by utilizing their unique, exponential, form $e^{-t/\sqrt{\epsilon}}$. The method has been employed in numerous fields in the natural sciences and has yielded a wealth of information on the leading order behavior of solutions to complicated ODEs and PDEs including nonlinear equations, equations with non-constant coefficients and boundary value problems. Though any particular application of the method is unique, the approach underlying the application is fairly uniform and it follows the general structure suggested by the onset of dissipation or dispersion when the small parameter, $\epsilon$, tends to 0. Both dissipation and dispersion are characterized by exponential behavior where the exponent is real in dispersion and imaginary in dissipation.

Thus, to demonstrate the general approach of the WKB method, consider the exponential form:

$$y(t) = A(t)e^{S(t)/\delta}, \quad \delta \to 0_+.$$

The phase, $S(t)$, is considered slowly varying in the breakdown region. As described above, when $S(t)$ is real, there is a boundary region of thickness $\delta$ in which $y(t)$ dissipates and when $S(t)$ is imaginary, there is a region of rapid oscillation (dissipation) where $y(t)$ has a period of order $\delta$. For constant $S(t)$ the solution is characterized by the slowly varying **amplitude**, $A(t)$.

These considerations lead to the search of solutions to a differential problem of the

general form of an exponential of a power series in a small parameter $\delta$:

$$y(t) \sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(t)\right), \quad \delta \to 0, \tag{5.1}$$

where $S_n(t)$ are the coefficient functions of the power series expansion and the small parameter $\delta$ is set by the particular form of the differential problem being solved. The following $2^{nd}$ order singular ODE exemplifies the application of the WKB method. The approximate solutions to the general (Schrödinger) equation

$$\epsilon^2 y'' = Q(t)y, \quad Q(t) \neq 0, \tag{5.2}$$

are easily found using the WKB method. Differentiation of the power series in equation (5.1) yields:

$$y' \sim \left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'\right)\exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n\right), \quad \delta \to 0 \tag{5.3}$$

and

$$y'' \sim \left(\frac{1}{\delta^2}\left(\sum_{n=0}^{\infty}\delta^n S_n'\right)^2 + \frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''\right)\exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n\right), \quad \delta \to 0. \tag{5.4}$$

Substituting these expressions in the differential equation (5.2) and dividing off the exponential factors yields:

$$\frac{\epsilon^2}{\delta^2}S_0'^2 + \frac{2\epsilon^2}{\delta}S_0'S_1' + \frac{\epsilon^2}{\delta}S_0'' + \ldots = Q(t). \tag{5.5}$$

Since $\delta$ is small, the largest term on the LHS is $\frac{\epsilon^2}{\delta^2}S_0'$ and since all other terms on the LHS are smaller than the first by order $\delta$ or higher, this leading term must balance the RHS (this is where the restriction $Q(t) \neq 0$ is employed!) so we must set $\delta = \epsilon$ in order for this term to balance $Q(t)$. Once the relation between $\delta$ and $\epsilon$ has been established, the terms on the LHS can be arranged in increasing powers of $\epsilon$ (or $\delta$) and the sequence of equations determines the coefficient functions $S_0, S_1, S_2, \ldots$ as follows:

$$O(\epsilon^0) : S_0'^2 = Q(t), \tag{5.6}$$

$$O(\epsilon^1) : 2S_0'S_1' + S_0'' = 0, \tag{5.7}$$

$$O(\epsilon^n) : 2S_0'S_n' + S_{n-1}'' + \sum_{j=1}^{n-1} S_j'S_{n-j}' = 0, \quad n \geq 2. \tag{5.8}$$

The two basic solutions of (5.6), called the eikonal equation, are:

$$S_0(t) = \pm\int_0^t \sqrt{Q(\tau)}d\tau. \tag{5.9}$$

Substitution of either of the $S_0(t)$ solutions in the linear equation (5.7), called the transport equation, yields the basic solution of $S_1(t)$:

$$S_1(t) = -\frac{1}{4}\ln Q(t). \tag{5.10}$$

The substitution of the two solutions of $S_0$ and the corresponding expression of $S_1$ in the power series expansion (5.1) then yields the approximate solution (to orders $\delta^{-1}$ and $\delta^0$ )

$$y(t) = c_1 Q^{-\frac{1}{4}}(t)\exp\left[\frac{1}{\epsilon}\int_{t_0}^{t}\sqrt{Q(\tau)}d\tau\right] + c_2 Q^{-\frac{1}{4}}(t)\exp\left[-\frac{1}{\epsilon}\int_{t_0}^{t}\sqrt{Q(\tau)}d\tau\right], \tag{5.11}$$

where $c_1$ and $c_2$ are the two constants of integration, determined from two associated initial or boundary conditions. Note that the solutions of $S_0$ and $S_1$ include only the fundamental solutions while the constants of integration are introduced in the expression of $y(t)$.

As demonstrated by this example, the general approach in the application of the WKB method consists of the following steps:

1. Expand the sought solution function in power series as in equation (5.1).
2. Set the relation between $\epsilon$ (from the equation) and $\delta$ (from the power series expansion) by requiring that the $O(1)$ terms in the equation balance the leading order term(s) where the two parameters appear i.e., term(s) of (lowest power of $\epsilon$) $\times$ (highest power of $1/\delta$).
3. Solve the sequence of equations for the coefficient functions $S_n(t)$ derived by balancing terms of order $\epsilon^n$.
4. Substituting the expressions of $S_n(t)$ in (5.2) and determining the constants of integration from the initial or boundary yields the approximate solution to the differential problem (equation and associated initial or boundary conditions).

In the remainder of this chapter examine several complex differential problems that exemplify the way the WKB method is applied in deriving approximate solutions to such problems. The derived analytic approximations are compared to direct numerical solutions of the equation, which demonstrates the unparalleled power of the WKB method in solving singular problems with rapid dispersion or dissipation.

## 5.1 Example 1: Solve $\epsilon^2 y'' = y$ $[y(0) = 1, y(1) = \exp(1/\epsilon)]$ using the WKB method

This is a boundary value problem in which the differential equation has the form of (5.2) with $Q(t) = 1$. The solution is detailed below by following the 4 steps outlined at the end of the previous subsection.

**Step 1.** Assume the solution has the general form $y(t) = \exp\left[\frac{1}{\delta}\left(S_0 + \delta S_1 + \delta^2 S_2 \ldots\right)\right]$ where $S_n(t)$ vary slowly with time.

**Step 2.** To establish the relation between $\delta$ and $\epsilon$, substitute this form of the solution in the differential equation i.e.,:

$$\epsilon^2 \frac{d^2}{dt^2} \exp\left[\frac{1}{\delta}\left(S_0 + \delta S_1 + \delta^2 S_2 \ldots\right)\right] = \exp\left[\frac{1}{\delta}\left(S_0 + \delta S_1 + \delta^2 S_2 \ldots\right)\right]. \qquad (5.12)$$

Carrying out the time differentiation and dividing off the common exponential coefficient then yields:

$$\frac{\epsilon^2}{\delta^2} S_0'^2 + \frac{2\epsilon^2}{\delta} S_0' S_1' + \frac{\epsilon^2}{\delta} S_0'' + \ldots = 1. \qquad (5.13)$$

The leading order term on the LHS is $\frac{\epsilon^2}{\delta^2}$ and for this term to balance the $O(1)$ term on the RHS, $\dfrac{\epsilon^2}{\delta^2}$ has to equal 1. Substituting $\epsilon = \delta$ then yields:

$$\begin{aligned} O(\epsilon^0) &: S_0'^2 \qquad\quad = 1, \\ O(\epsilon^1) &: 2S_0' S_1' + S_0'' = 0, \end{aligned}$$

**Step 3.** Solving the above equations for $S_0$ and $S_1$ yields:

$$\begin{aligned} O(\epsilon^0) &: S_0(t) \qquad = \pm t \\ O(\epsilon^1) &: 2AS_1' + 0 = 0 \implies S_1(t) = \text{constant} \end{aligned}$$

**Step 4.** Substituting $\delta = \epsilon$, $S_0(t) = t$ and $S_1(t) = $ constant into the solution $y(t) = \exp\left[\frac{1}{\delta}\left(S_0 + \delta S_1 + \ldots\right)\right]$ yields: $y_+(t) = \exp\left[\left(\frac{t}{\epsilon}\right) + c\right]$. Similarly, we can obtain $y_-(t) = \exp\left[\left(\frac{-t}{\epsilon}\right) + c\right]$. Assuming that the general solution is a linear combination of $y_+(t)$ and $y_-(t)$ i.e., $y(t) = c_1 y_+(t) + c_2 y_-(t)$ and applying the boundary conditions $y(0) = 1$ and $y(1) = \exp(1/\epsilon)$ yields $y(t) = \exp\left(\frac{t}{\epsilon}\right)$ which is the exact solution of the linear, constant coefficient equation, differential eigenvalue problem. In this simple example, the WKB method yielded the exact solution of the problem! A somewhat more complicated problem will be solved in the next example and its WKB approximate solution will be compared to a numerical solution of the problem.

## 5.2 Example 2: Solve $\epsilon^2 y'' = (1+t^2)^2 y$ $[y(0) = 0, y'(0) = 1]$ using the WKB method

This is an initial value problem in which the differential equation has the form of (5.2) with $Q(t) = (1+t^2)^2$. The solution is detailed below by following the 4 steps outlined at the end of the previous subsection.

**Step 1.** Assume the solution has the general form $y(t) = \exp\left[\frac{1}{\delta}\left(S_0 + \delta S_1 + \delta^2 S_2 \ldots\right)\right]$ where $S_n(t)$ vary slowly with time.

**Step 2.** To establish the relation between $\delta$ and $\epsilon$, substitute this form of the solution in the differential equation i.e.,:

$$\epsilon^2 \frac{d^2}{dt^2} \exp\left[\frac{1}{\delta}\left(S_0 + \delta S_1 + \delta^2 S_2 \ldots\right)\right] = (1+t^2)^2 \exp\left[\frac{1}{\delta}\left(S_0 + \delta S_1 + \delta^2 S_2 \ldots\right)\right]. \tag{5.14}$$

Carrying out the time differentiation and dividing off the common exponential coefficient then yields:

$$\frac{\epsilon^2}{\delta^2} S_0'^2 + \frac{2\epsilon^2}{\delta} S_0' S_1' + \frac{\epsilon^2}{\delta} S_0'' + \ldots = (1+t^2)^2. \tag{5.15}$$

The leading order term on the LHS is $\dfrac{\epsilon^2}{\delta^2}$ and for this term to balance the $O(1)$ term on the RHS, $\dfrac{\epsilon^2}{\delta^2}$ has to equal 1. Substituting $\epsilon = \delta$ then yields:

$$\begin{aligned} O(\epsilon^0) &: S_0'^2 &&= (1+t^2)^2, \\ O(\epsilon^1) &: 2S_0' S_1' + S_0'' &&= 0, \end{aligned}$$

**Step 3.** Solving the above equations for $S_0$ and $S_1$ yields:

$$\begin{aligned} &O(\epsilon^0) : S_0(t) && = \pm\left(t + \frac{t^3}{3}\right) \\ &O(\epsilon^1) : 2(1+t^2)S_1' + 2t = 0 \implies S_1(t) = -\frac{1}{2}\log(1+t^2) \end{aligned}$$

**Step 4.** Substituting $\delta = \epsilon$, $S_0(t) = \pm\left(t + \dfrac{t^3}{3}\right)$ and $S_1(t) = -\dfrac{1}{2}\log(1+t^2)$ into the general form $y(t) = \exp\left[\dfrac{1}{\delta}\left(S_0 + \delta S_1 + \ldots\right)\right]$ yields the two solutions $y_+(t) = \dfrac{1}{\sqrt{1+t^2}} \exp\left[\dfrac{1}{\epsilon}\left(t + \dfrac{t^3}{3}\right)\right]$ and $y_-(t) = \dfrac{1}{\sqrt{1+t^2}} \exp\left[-\dfrac{1}{\epsilon}\left(t + \dfrac{t^3}{3}\right)\right]$. The resulting general solution of $y(t)$, which the linear combination of the two basic forms, $y_+(t)$ and $y_-(t)$ is:

$$\begin{aligned} y(t) &= c_1 y_+(t) + c_2 y_-(t) \\ &= \frac{1}{\sqrt{1+t^2}}\left(c_1 \exp\left[\frac{1}{\epsilon}\left(t + \frac{t^3}{3}\right)\right] + c_2 \exp\left[-\frac{1}{\epsilon}\left(t + \frac{t^3}{3}\right)\right]\right). \end{aligned}$$

Applying the initial condition $y(0) = 0$ yields $c_1 + c_2 = 0 \implies c_1 = -c_2$. Therefore, $y(t)$ can be rewritten as: $y(t) = \dfrac{2c_1}{\sqrt{1+t^2}} \sinh\left[\dfrac{1}{\epsilon}\left(t + \dfrac{t^3}{3}\right)\right]$ (recall: $e^x - e^{-x} = 2\sinh(x)$). Applying the other initial conditions, $y'(0) = 1$, yields $\dfrac{2c_1}{\epsilon} = 1 \implies 2c_1 = \epsilon$, so the particular solution that satisfies both initial conditions is: $y(t) = \dfrac{\epsilon}{\sqrt{1+t^2}} \sinh\left[\dfrac{1}{\epsilon}\left(t + \dfrac{t^3}{3}\right)\right]$.

There is an excellent agreement between the solution to the initial value problem obtained using numerical integration and the leading order WKB approximation (the reader should verify this as an exercise!). Fig. 5.1 shows the relative error between the numerical solution and the WKB approximation for different values of $\epsilon$. Even for $\epsilon = 0.4 \gg 0$ the magnitude of the maximum relative error between solutions is $< 5\% < O(0.4^2)$ (the dotted-dashed green line, Fig. 5.1).

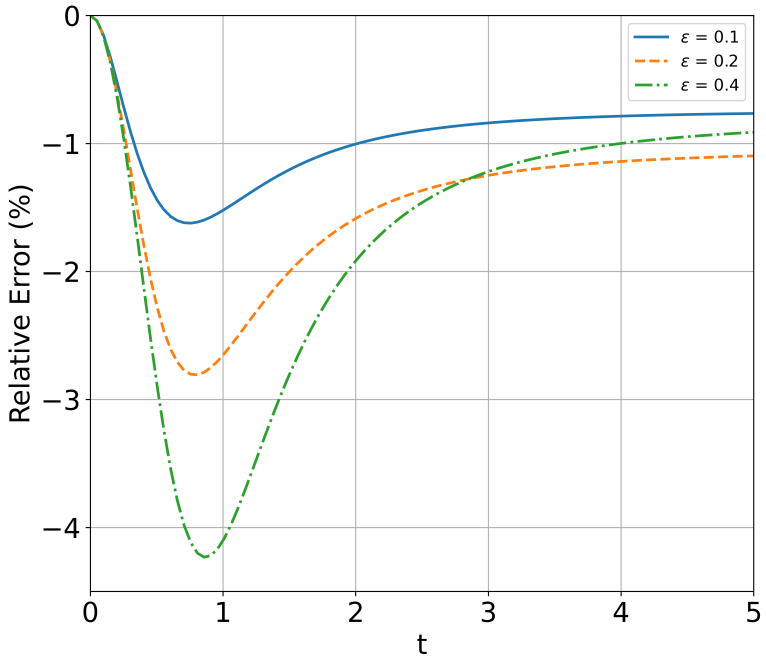


Figure 5.1: Relative error between the numerically obtained solution and the leading order WKB approximation for differnt values of $\epsilon$.

## 5.3 Homework assignment 7

Find the WKB solutions of the following equation:

$$\epsilon y'' + (c - \epsilon)y' - ay = 0$$

where $a$ and $c$ are constants. For different values of $a$ and $c$, compare the numerical solution to the equation with the WKB solutions and determine the values of $\epsilon$ for which the WKB solutions are good approximations (i.e., the relative error is $< O(\epsilon^2)$).

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